Thermodynamic formalisms for Markov subshifts on d-trees

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Ising model on trees

Consider an ideal magnet with a d-tree lattice structure $T^{(d)}$ as follows.

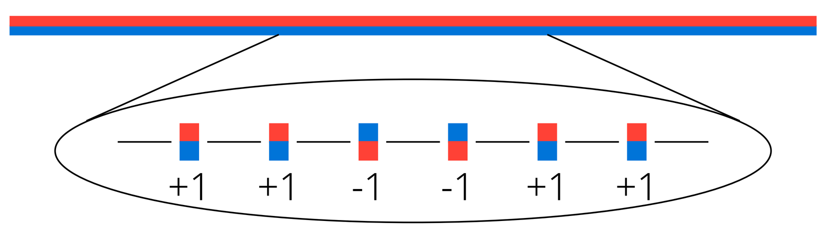


Figure 1: $d = 1$

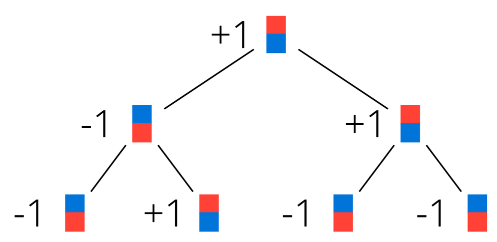


Figure 2: $d = 2$

- The orientations of the spins are governed by a **state** μ, a probability measure over all possible **configurations** $\mathcal{A}^{T^{(d)}} := \{+1,-1\}^{T^{(d)}}$.
- The **free energy** of μ restricted to initial n-subtree Δ_n is

$$F_n(\mu) = \int \varphi_n d\mu - h_n(\mu) \quad (\beta > 0),$$

where, for some constant $a, b \in \mathbb{R}$,

(entropy) $$h_n(\mu) = \sum_{w \in \mathcal{A}^{\Delta_n}} -\mu[w] \log \mu[w],$$

(internal energy) $$\varphi_n(x) = \sum_{g \in \Delta_n \setminus \{\mathrm{root}\}} a x_{\varsigma(g)} x_g + \sum_{g \in \Delta_n} b x_g,$$

with $\varsigma(g)$ denoting the parent of g.

- An **equilibrium state** is a state μ minimizing the free energy per site

$$F(\mu) = \limsup_{n\to\infty} \frac{1}{n} F_n(\mu).$$

The equilibrium state is the one that could be observed macroscopically.

Properties

($d = 1$) There is a unique invariant equilibrium μ, which is Markov and

$$F(\mu) = -P \quad \text{with } P := \limsup_{n\to\infty} \frac{1}{|\Delta_n|} \log \sum_{w \in \mathcal{A}^{\Delta_n}} \sup_{x \in [w]} e^{-\varphi_n(x)}.$$

Moreover, $\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) + \lim_{n\to\infty} \frac{1}{n} \log \mu\left[x_{\Delta_n}\right] = -P$ for μ-a.e. x.

($d \geq 2$) R. M. Burton, C.-E. Pfister, and J. E. Steif [1] showed

$$\inf\{F(\mu) : \mu \text{ invariant}\} > -P \text{ iff } (a, b) \neq (0, 0).$$

Question

- *Can we identify the equilibrium state (without invariance assumption)?*
- *For Markov measures, is there a pointwise convergence?*

Setting

- The questions are studied under the following setting. Let $\mathcal{A}$ be a finite set and $A \in R_{\geq 0}^{\mathcal{A}\times\mathcal{A}}$. Assume the system is defined by

(configurations) $$\mathcal{T}_A = \left\{x \in \mathcal{A}^{T^{(d)}} : A_{x_g, x_{\varsigma(g)}} > 0, \forall g \in T^{(d)} \setminus \{\mathrm{root}\}\right\}$$

(internal energy) $$\varphi_n(x) = -\textstyle\sum_{g \in \Delta_n \setminus \{\mathrm{root}\}} \log A_{g, \varsigma(g)}$$

with an additional assumption

(irreducibility) $$\forall a, b \in \mathcal{A}, \exists n \in \mathbb{N} \text{ such that } (A^n)_{a,b} > 0.$$

- For simplicity, define $\Psi_{A,d} : \mathbb{R}^{\mathcal{A}}_{\geq 0} \to\in \mathbb{R}^{\mathcal{A}}_{\geq 0}$ as $\Psi_{A,d}(u) = (A^T u)^d$.

Result 1: equilibrium states

Theorem (J.-C. Ban and Y.-L. Wu [2])

There is a layer-dependent Markov equilibrium μ with

$$F(\mu) = -P \text{ with } P := \limsup_{n\to\infty} \frac{1}{|\Delta_n|} \log \sum_{w \in \mathcal{A}^{\Delta_n}} \sup_{x \in \mathcal{T}_A \cap [w]} e^{-\varphi_n(x)}.$$

Moreover, $\limsup_{n\to\infty} \frac{1}{|\Delta_n|}\left(\varphi_n(x) + \log \mu\left[x_{\Delta_n}\right]\right) = -P$ for μ-a.e. x.

Note: The equilibrium μ found above is not unique.

Result 2: pointwise convergence

Remark

- A is irreducible iff there exists a partition $(\mathcal{A}_j)_{j=0}^{p-1}$ of $\mathcal{A}$ such that

$$a \in \mathcal{A}_i, b \in \mathcal{A}_j \quad \text{if and only if} \quad \left(A^{pn+j-i}\right)_{a,b} > 0 \text{ for all large } n \in \mathbb{N}.$$

- If μ is a Markov measure with transition matrix M and $A = M$, then

$$\varphi_n(x) = \log \mu\left(\left[x_{\Delta_n}\right] \middle| x_{\mathrm{root}}\right).$$

Therefore, it suffices to study merely $\frac{1}{|\Delta_n|}\varphi_n(x)$.

Theorem (J.-C. Ban, G.-Y. Lai, and Y.-L. Wu [3])

Suppose μ is a invariant Markov measure with (irreducible) transition matrix M and $\mathrm{supp}(\mu) = \mathcal{T}_A$. If $a \in \mathcal{A}_0$, then for every interval $I \subset \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{|\Delta_{pn+j}|} \log \mu\left(\frac{\log \varphi_{pn+j}(x)}{|\Delta_{pn+j}|} \in I \,\middle|\, x_{\mathrm{root}} = a_0\right) = \sup_{\alpha \in I} \Lambda_j^*(\alpha)$$

where $\Lambda_j^*(\alpha) = \sup_{\mu\in\mathbb{R}} \mu\alpha - \lim_{n\to\infty} \frac{1}{|\Delta_n|} \log\left\|\Psi^{pn}_{A^\mu \odot M, d}\left(1_{\mathcal{A}_j}\right)\right\|$. In particular, for μ-a.e. x satisfying $x_{\mathrm{root}} = a_0$,

$$\lim_{n\to\infty} \frac{\varphi_{pn+j}(x)}{|\Delta_{pn+j}|} = \alpha_j := \mathbb{E}\left(\frac{\varphi_p(y)}{|\Delta_p \setminus \{\mathrm{root}\}|} \,\middle|\, \{y : y_{\mathrm{root}} \in \mathcal{A}_j\}\right).$$

Application

- By introducing a metric $D(x, y) = e^{\sup\{-|\Delta_n| : x_{\Delta_n} = y_{\Delta_n}\}}$ for $\mathcal{T}_A$, we have the following theorem.

Corollary

Let $A \in \{0,1\}^{\mathcal{A}\times\mathcal{A}}$ be irreducible and $\mathcal{R}_{p,d} = \left\{r \in (0, d]^p : \prod_{i=0}^{p-1} r_i = 1\right\}$.

$$d^{-1}\,\overline{\dim}_B \mathcal{T}_A = \underline{\dim}_B \mathcal{T}_A = \dim_P \mathcal{T}_A = \max_\mu \dim_P \mu = \lim_{n\to\infty} \frac{\left\|\Psi^n_{A,d}(1)\right\|}{|\Delta_n|}$$

$$\dim_H \mathcal{T}_A = \max_\mu \dim_H \mu = \min_{r \in \mathcal{R}_{p,d}} \left(\sum_{\ell=0}^{p-1} \prod_{i=0}^{\ell} r_i^{-1}\right)^{-1} \cdot \log \rho\left(\mathcal{L}_{A,r}\right)$$

where $\mathcal{L}_{A,r} := \Psi_{A,r_{p-1}} \circ \cdots \circ \Psi_{A,r_0}$ with

$$\rho\left(\mathcal{L}_{A,r}\right) = \sup\left\{\alpha \in \mathbb{R} : \mathcal{L}_{A,r}(u) = \alpha u \in \mathbb{R}^{\mathcal{A}}_{\geq 0} \setminus \{0\}\right\}.$$

Figures

Figure 3: $\Lambda_j^*(\alpha)$

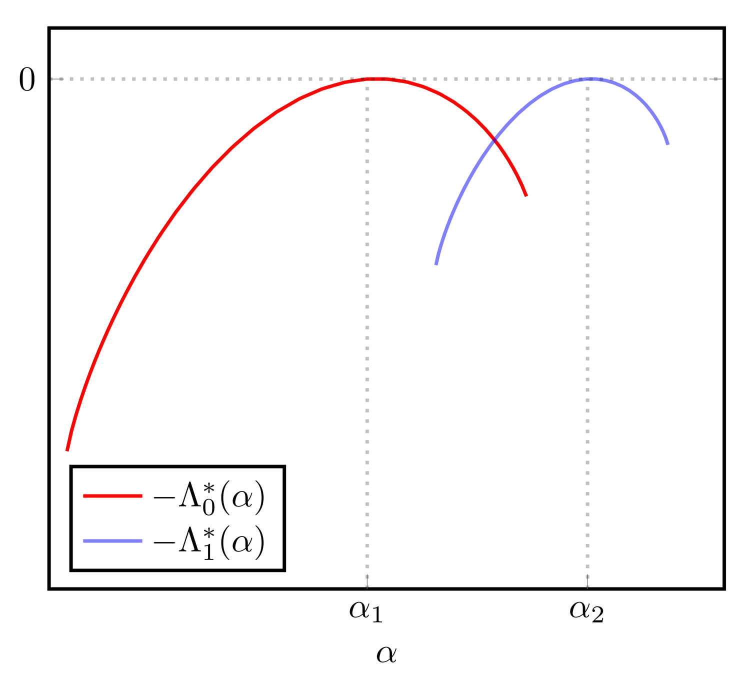
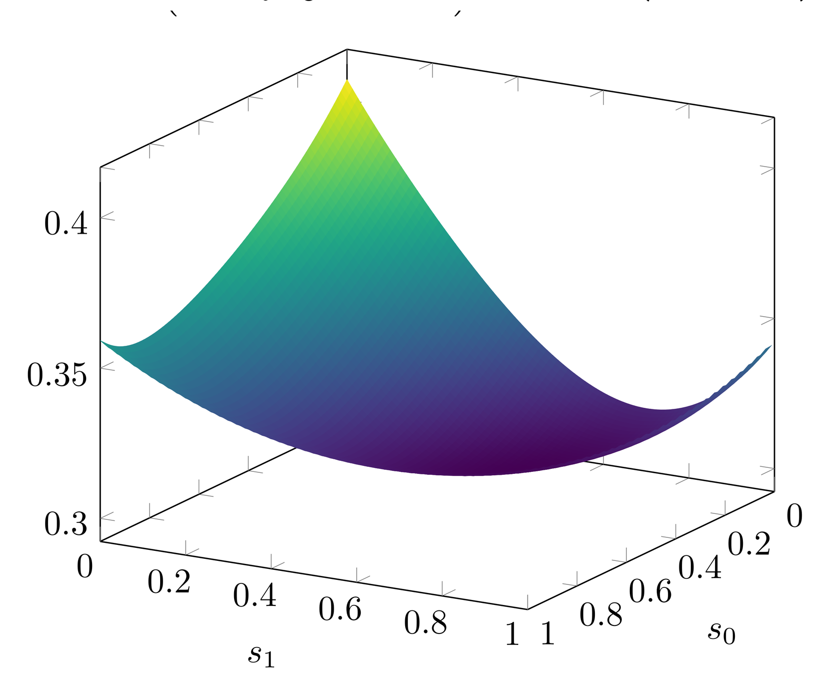


Figure 4: $\prod_{i=0}^{\ell} r_i^{-1}(s))^{-1} \log \rho\left(\mathcal{L}_{A,r(s)}\right)$



Sources

[1] R. M. Burton, C.-E. Pfister, and J. E. Steif, "The variational principle for Gibbs states fails on trees," *Markov Processes And Related Fields*, vol. 1, no. 3, pp. 387–406, 1995.

[2] J.-C. Ban and Y.-L. Wu, "On the topological pressure of axial product on trees," *arXiv:2310.10242*.

[3] J.-C. Ban, G.-Y. Lai, and Y.-L. Wu, "Hausdorff dimensions of topologically transitive Markov hom tree-shifts," *arXiv:2401.05320*.