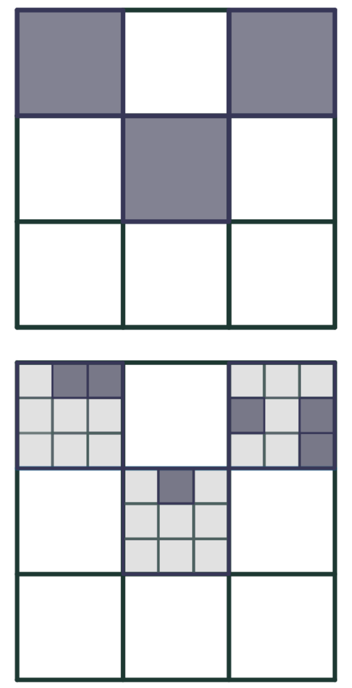
# Overlapping Mandelbrot percolation

*Vilma Orgoványi*[1,2] *and Károly Simon*[1,2]

## Abstract.

Using a similar random process to the one which yields the Mandelbrot percolation sets, we can define overlapping Mandelbrot percolation sets. We investigate these sets from the viewpoint of positivity of Lebesgue measure, existence of interior points.

## Fractal percolation.



The (homogeneous) fractal percolation set $\Lambda_p = \Lambda^{(d)}_{(M,p)}$ is a two-parameter $(M, p)$ family of random fractals in $\mathbb{R}^d$. For $M = 3$ and $d = 2$, the construction is as follows. Start with the unit square.

I Subdivide the square into $M^d = 3^2$ congruent subsquares.

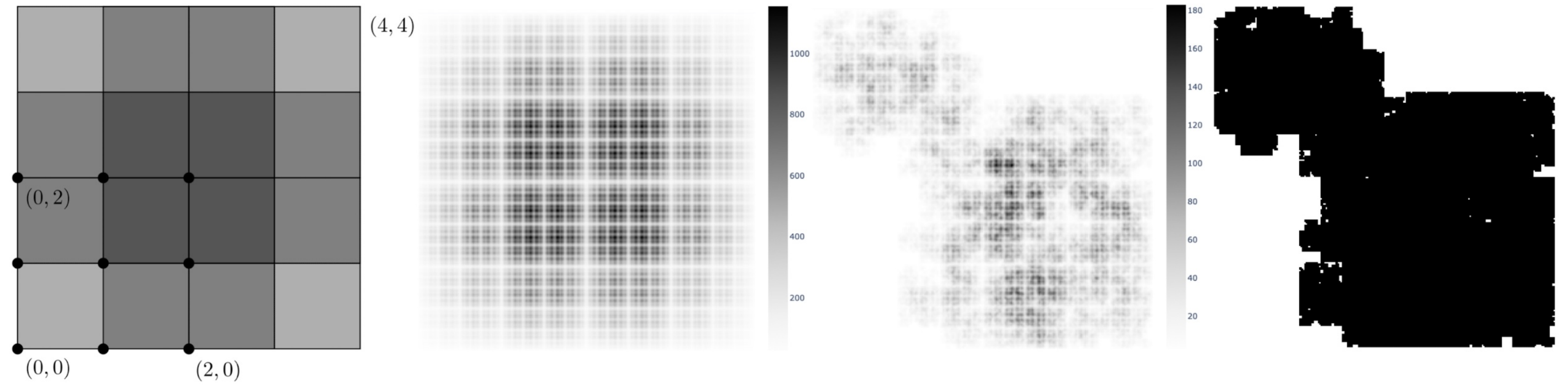
II Each of these are retained with probability $p$ and discarded with probability $1 - p$, independently of everything.

Repeat these steps in every retained square independently ad infinitum or until we do not have any retained squares left (which event is called called extinction). The resulting set is $\Lambda_p$.

## Integer overlapping Mandelbrot percolations.

For the integer overlapping Mandelbrot percolation we run an analogous process on an IFS that can contain certain*, non-negligible overlaps.

Example: Randomized tartan, $(\{\mathbf{x}/2 + \mathbf{t}_i\}_{i=1}^{9}$ where $\mathbf{t}_i$ runs through the set $\{0, 1, 2\}^2)$.




The first level cylinders, with lower-left vertex denoted; the sixth level approximation of the deterministic set and a realization of the randomized set (with parameter $p = 0.7$) where darkness indicates the number of cylinders and the retained cylinders.

## *Details.

Run the percolation on the cylinders of IFSs of the following form: $\mathcal{S} := \left\{S_i(x) := \frac{1}{L}\mathbf{x} + \mathbf{t}_i\right\}_{i=0}^{M-1}$, $S_i : \mathbb{R}^d \to \mathbb{R}^d$, where: $L \in \mathbb{N} \setminus \{0, 1\}$, $\mathbf{t}_i \in \mathbb{N}^d$, $\exists h \in \mathbb{N}$, $L - 1 | h$, $\forall j \in [d]$ $0 = \min_i \mathbf{t}_i(j)$, $h = \max_i \mathbf{t}_i(j)$.

## Preliminaries.

For the $d$-dimensional deterministic system

$$\mathcal{S} = \left\{S_i(\mathbf{x}) = \frac{1}{L}\mathbf{x} + \mathbf{t}_i\right\}_{i=1}^{K},$$

we can associate a set of non-negative matrices $\mathcal{M} := \{\mathbf{M}_1, \ldots, \mathbf{M}_{L^d}\}$.
For the randomized tartan example these are as follows:

$$\mathbf{M}_1 = \begin{bmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&0&1&0\\ 1&1&1&1 \end{bmatrix}, \mathbf{M}_2 = \begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 1&1&1&1\\ 0&1&0&1 \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} 1&0&1&0\\ 1&1&1&1\\ 0&0&1&0\\ 0&0&1&1 \end{bmatrix}, \mathbf{M}_4 = \begin{bmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}.$$

## Results.

Consider the randomized IFS $\mathcal{S}$, with attractor $\Lambda_p$ and the associated set of matrices $\mathcal{M} = \{\mathbf{M}_1, \ldots, \mathbf{M}_L\}$. We assume, that

- each matrix has a positive element in every row and column,
- there exists a product $\mathbf{M}_{i_1} \cdots \mathbf{M}_{i_k}$ which is a strictly positive.

Then

1. $\Lambda_p$ has positive Lebesgue measure almost surely conditioned on non-extinction iff $p > \mathrm{e}^{-\lambda}$;
2. if $p < e^{-\rho}$, $\Lambda_p$ has empty interior almost surely.

Here $\lambda$ is the Lyapunov exponent with respect to the uniform measure, $\rho$ is the logarithm of the lower spectral radius corresponding to $\mathcal{M}$.
($\lambda$ is the a.s. value of $\lim_{n\to\infty} \frac{1}{n} \log \|M_{i_1} \cdots M_{i_n}\|$, and $\rho = \lim_{n\to\infty} 1/n \log \min\{\|M_{i_1} \ldots M_{i_n}\|, i_1, \cdots, i_n \in \{1, \ldots, L\}^n\}$.)

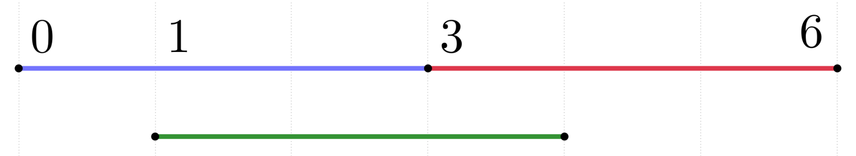
## Application to the randomized tartan example.

- When $p > 0.7712$ then the set has positive two dimensional Lebesgue measure almost surely conditioned on non-extinction.
- $\Lambda_p$ has empty interior almost surely if $p < 1$.

Remark. When $p > 0.993$ then by [1] the set contains a curve which connects the left and right walls with positive probability.

## Randomized 0-1-3, with contraction ratio $1/2$.

$\Lambda_p$ is the percolation on the cylinders of the IFS $\{1/2x, 1/2x + 1, 1/2x + 3\}$.




Associated matrices:

$$\mathbf{M}_0 = \begin{bmatrix} 1&0&0\\ 0&1&1\\ 0&1&0 \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 1&1&0\\ 1&0&1\\ 0&0&1 \end{bmatrix}.$$

### Results:

There exist parameters $2/3 < p_0 < p_1 < 1$ such that

- for $p_0 < p < p_1$, $\Lambda_p$ has positive Lebesgue measure almost surely conditioned on non-extinction but empty interior almost surely,
- for $p_1 < p \le 1$, $\Lambda_p$ has non-empty interior almost surely conditioned on non-extinction.

## Further information.

Our model is a special case of the more general random percolation self-similar Cantor sets introduced by Falconer and Jin in [2, Section 6]. The poster is based on our preprint:

## References.

[1] Henk Don.
*Reflecting Walls and Dissipating Tiles: Billiards and Fractal Percolation.*
PhD thesis.

[2] Kenneth J Falconer and Xiong Jin.
Exact dimensionality and projections of random self-similar measures and sets.
*Journal of the London Mathematical Society*, 90(2):388–412, 2014.

## Affiliation.

[1]Department of Stochastics, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
[2]HUN-REN-BME Stochastics Research Group, Műegyetem rkp. 3., H-1111 Budapest, Hungary 1, 1111 Budapest, Hungary