Ergodic Theorem for Iterated Function System with Local Radial Contractions

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Objectives

Our primary objective is to study the following topics:

- The existence and uniqueness of an invariant measure associated with the Markov operator induced by an IFSp with local radial contractions.
- To study an ergodic theorem for this invariant measure.

1. Introduction

The study of invariant measures has been at the center of the theory of iterated function systems (IFS) since its introduction by Hutchinson [4]. An invariant measure for an IFS is a probability measure that remains unchanged under the action of the system's mappings. More formally, if $\{w_i\}_{i=1}^N$ is a collection of contraction mappings on a space X, and $\{p_i\}_{i=1}^N$ is a corresponding set of probabilities with $p_i > 0$ and $\Sigma_{i=1}^N p_i = 1$, then a measure μ is called invariant if it satisfies the equation:

$$\mu(B) = \sum_{i=1}^{N} p_i \mu(w_i^{-1}(B))$$

for all measurable sets B.

Ergodicity plays a crucial role in analyzing and controlling chaotic systems modeled by IFS. Elton [2] established the first fundamental result on the ergodicity of the invariant measure corresponding to a hyperbolic IFS. This is an IFS counterpart to the traditional Birkhoff ergodic theorem for a single map [5]. The current work aims to establish a theorem for a more general IFS, namely, an IFS consisting of local radial contractions.

2. Local Radial Contraction

- (X, d) be a compact metric space.
- $f : X \to X$ is a *Local Radial Contraction* if there exists a $\lambda \in [0, 1)$ such that for all $x_0 \in X$, there is $\varepsilon_{x_0} > 0$ satisfying the following:
$$d\big(f(x_0), f(y)\big) < \lambda\, d(x_0, y),$$
for $y \in X$ with $d(x_0, y) < \varepsilon_{x_0}$.

If X is a compact convex set, then a Local Radial Contraction is a contraction in the whole space X.

3. Example of an LCR

Let $X \coloneqq \{(\cos\theta, \sin\theta) : \ 0 \le \theta \le \frac{3\pi}{2}\}$ be endowed with the Euclidean metric on $\mathbb{R}^2$. Consider the maps $f : X \to X$ defined by

$$f(\cos\theta, \sin\theta) = (\cos\frac{\theta}{3}, \sin\frac{\theta}{3}).$$

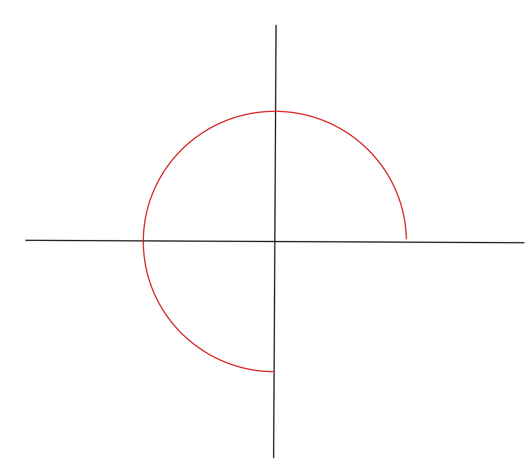

Figure 1: The set X

4. IFSp with LRCs

- Let $\{w_i\}_{i=1}^N$ be a set of LRCs.
- The triple $\{X; w_i; p_i : i = 1, \dots, N\}$ is called an IFS with probabilities consisting of LRCs.
- $C(X)^*$ is the space of all bounded linear functionals on $C(X)$ or the space of all Borel regular measures on X.
- The operator $M : C(X)^* \to C(X)^*$ defined as
$$M\nu \coloneqq \sum_{i=1}^{N} p_i \nu \circ w_i^{-1},$$
is called the **Markov Operator**
- The fixed point under this operator is known as the **Invariant Measure** associated with the IFSp.

Existence of Invariant Measure

The Markov operator $M : C(X)^* \to C(X)^*$ corresponding to the local radial contractive IFS with probabilities, has a unique fixed point μ in $C(X)^*$.

5. Ergodic Transformation

A measure preserving transformation (mpt) is a quartet $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a measure space, and

1. T is measurable:
$E \in \mathcal{B} \implies T^{-1}(E) \in \mathcal{B}$.
2. μ is T–invariant:
$\mu(T^{-1}(E)) = \mu(E)$ for all $E \in \mathcal{B}$.

An mpt $T : X \to X$ is called ergodic if $T^{-1}(E) = E$ implies $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Example: The one sided shift operator τ on the symbolic space $\Omega = \{1, \dots, N\}^{\mathbb{N}}$ defined as $\tau(\sigma_1, \sigma_2, \sigma_3, \dots) = (\sigma_2, \sigma_3, \dots)$ is Ergodic.

Ergodic Theorem for IFS with LRCs

Suppose X is a compact metric space. For any continuous function $f : X \to \mathbb{R}$ and any $x \in X$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i \le n} f(w_{\sigma^i}(x)) = \int_X f(y)\, \mathrm{d}\mu(y),$$

for P almost all $\sigma \in \Omega$.

6. Proof Outline

- The proof is based on the Banach limit technique.
- Since $\left(\frac{1}{n}\Sigma_{i\le n} f\big(w_{\sigma^i}(x)\big)\right)_{n=1}^{\infty}$ is bounded sequence, we use an arbitrary Banach limit $\phi : l^\infty \to \mathbb{R}$ to define a linear functional Φ on $C(X)$ such that
$$\Phi(f) = \left(\frac{1}{n}\sum_{i\le n} f\big(w_{\sigma^i}(x)\big)\right)_{n=1}^{\infty}$$
- Using the following property
$\lim_{n\to\infty} d(w_{\sigma_n} \circ \dots \circ w_{\sigma_1}(x),$
$w_{\sigma_n} \circ \dots \circ w_{\sigma_2}(x)) = 0,$
and the fact that the one sided shift map τ is ergodic we show that $\phi\Big((\frac{1}{n}\Sigma_{i\le n} f\big(w_{\sigma^i}(x)\big))_{n=1}^{\infty}\Big)$ is constant for P almost all σ.
- Φ is well defined and therefore we get a Borel regular measure μ_Φ on X such that
$$\phi\Big((\frac{1}{n}\sum_{i\le n} f(w_{\sigma^i}(x)))_{n=1}^{\infty}\Big) = \int_X f \mathrm{d}\mu_\Phi,$$
P almost all σ.
- It is easy to see that μ_Φ is a probability measure. We show that μ_Φ is an invariant measure under M.

Proof Outline (Cont.)

- Since μ is unique and ϕ was arbitrary, for P almost all σ
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i\le n} f(w_{\sigma^i}(x)) = \int_X f(y)\, \mathrm{d}\mu(y),$$
for all f and x. This completes the proof of the Ergodic theorem.

7. Conclusion and Further Directions

We investigate the existence and uniqueness of the invariant measure for an IFS with local radial contractions and establish an ergodic theorem for this IFS. Our proof adapts a technique based on [1, 3], which establishes the ergodic theorem for hyperbolic and weakly hyperbolic IFSs. Inspired by similar studies, our future work will identify conditions on the maps and invariant measure that ensure the ergodic theorem's conclusion.

8. References

[1] A. Arbieto, A. Junqueira, B. Santiago, On weakly hyperbolic iterated function systems. Bull. Braz. Math. Soc., 48 (2017), 111-140.

[2] H. Elton, An ergodic theorem for iterated maps, Ergod. Th. Dyn. Syst., 7 (1987), 481-488.

[3] B. Forte, F. Mendivil, A classical ergodic property for IFS: A simple proof, Ergod. Theory Dyn. Syst., 18(3), (1998), 609 - 611.

[4] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., 30, (1981), 713-747.

[5] P. Walter, An introduction to ergodic theory (Vol. 79). Springer Science & Business Media, (2000).

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