# New Hausdorff type dimensions and optimal bounds for bilipschitz invariant dimensions

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- Generalised intermediate dimensions (Banaji, 2023).

# Self-similar sets and similarity dimensions

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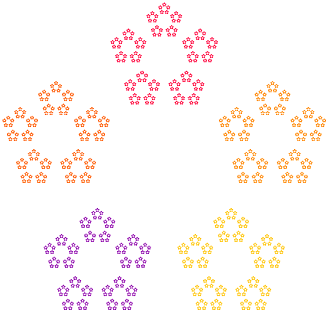

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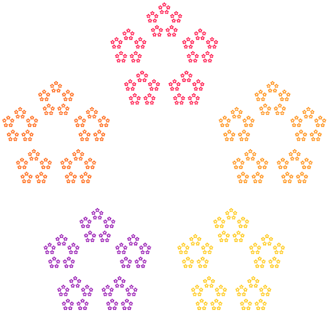

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All the fractal dimensions of these sets agree with their similarity dimension.

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- bilipschitz invariance:

  $\dim A = \dim B$ if $A$ and $B$ are bilipschitz equivalent.

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  The others can be modified to force this propery; in fact, this is a possible way to obtain packing dimension $\dim_{\mathrm{P}}$ from the upper box dimension $\overline{\dim}_{\mathrm{B}}$.

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Much more ambitous version:

**Problem (Fraser)**

*Find a list of natural poperties of dimensions that uniquely characterize the Hausdorff dimension.*

# Optimal bounds for "reasonable" dimensions

**Theorem (Balka-K 2023 Spring)**

*Let $D$ be a function defined on the compact subsets of $\mathbb{R}^n$. If $D$ is monotone, Lipschitz stable and it agrees with the similarity dimension for homogenous SSC self-similar sets then for any compact $K \subset \mathbb{R}^n$*

$$\dim_{\mathrm{H}}(K) \leq D(K) \leq \overline{\dim}_{\mathrm{B}}(K).$$

*Furthermore, if $D$ is also $\sigma$-stable then*

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Note that the Assouad dimension $\dim_{\mathrm{A}}$ can be greater than but it is not Lipschitz stable, only bilipschitz invariant.

# Rutar's question

**Theorem (Balka-K 2023 Spring, shortly stated again)**

*If $D$ is monotone, Lipschitz stable and it agrees with the similarity dimension for the simplest self-similar sets then on compact sets of $\mathbb{R}^n$ we have $\dim_H \leq D \leq \overline{\dim}_B$ .*
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and $\dim_{ML}$ is the optimal lower bound for any $n$.

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$$\dim_{\mathrm{ML}}(K) \leq D(K).$$

*If $n = 1$ then for any compact $K \subset \mathbb{R}^n$ we also have*

$$D(K) \leq \dim_{\mathrm{A}}(K),$$

*but this latter statement is false when $n > 1$.*

Counter-example for $D(K) \leq \dim_{\mathrm{A}}(K)$ when $n > 1$:

$$D(K) = \begin{cases} \dim_{\mathrm{A}} K & \text{if K is totally disconnected,} \\ n & \text{otherwise.} \end{cases}$$

Then $D(L) = n > 1 = \dim_{\mathrm{A}}(L)$ for any closed line segment.

# Fraser's question

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*Find a list of natural poperties of dimensions that uniquely characterize the Hausdorff dimension.*

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$\dim_H : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ *is a Borel function, but* $\dim_P : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ *is not.*

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Question of Fraser: $D : \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}$ is Borel, monotone, Lipschitz stable, $\sigma$-stable and it agrees with the similarity dimension for the simplest self-similar sets $\Longrightarrow D = \dim_H$ ?

# A new Hausdorff type dimension

Recall that for any $E \subset \mathbb{R}^n$,
$\dim_H(E) = \inf \left\{ s \geq 0 \colon \forall \varepsilon > 0 \ \exists (E_i) \ E \subset \cup E_i, \sum_{i=1}^{\infty} |E_i|^s < \varepsilon \right\}.$

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## Definition ($D$-diameter restricted Hausdorff dimension)

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where $D \subset (0, \infty)$ is a given set of allowed diameters.

To make things nicer we require $\inf D = 0$ and also that for any $x \in D$ there exists $\delta > 0$ such that $[x, x + \delta) \subset D$.

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**Theorem (Balka-K 2024 Autumn)**

*The above way we obtain continuum many distinct dimensions and each of them has all the properties we have listed so far.*

This gives a negative answer to Fraser's question.

# New hope

Banaji suggested that perhaps Fraser's question can be saved by requiring the following additional property:

Hölder stability: $\dim f(A) \leq \frac{1}{\alpha} \dim A$ if $\alpha \in (0, 1]$ and $f$ is $\alpha$-Hölder.

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*Hausdorff dimension is not the only dimension that has all the properties we have mentioned so far.*

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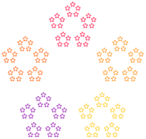

← a fractal under the midnight sun