1-Lipschitz maps onto polygons

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Kolmogorov's question

Question (Kolmogorov, 1932)

Let $A \subseteq \mathbb{R}^2$ be a measurable set with $\lambda(A) < \infty$, and let $\varepsilon > 0$. Does there exist a 1-Lipschitz map $f : A \to \mathbb{R}^2$ such that $\lambda(f(A)) \geq \lambda(A) - \varepsilon$ and $f(A)$ is a polygon?

Definition

A set $A \subseteq \mathbb{R}^2$ is a **polygon** if its boundary ∂A can be covered by finitely many lines.

- Balka, Elekes, Máthé (2013): False in general
- Open for compact sets

The compact case

Let $K \subseteq \mathbb{R}^2$ be a compact set.
$\mathbf{Q1}(K)$: For every $\varepsilon > 0$, there is a 1-Lipschitz map $f : K \to \mathbb{R}^2$ such that $f(K)$ is a polygon and $\lambda(f(K)) \geq \lambda(K) - \varepsilon$.
$\mathbf{Q2}(K)$: For every $\varepsilon > 0$, there is a 1-Lipschitz map $f : K \to \mathbb{R}^2$ such that $f(K)$ is a polygon and $|f(x) - x| \leq \varepsilon$ for every $x \in K$.

Theorem (Balka, Elekes, Máthé)

$(\forall K\ \mathbf{Q1}(K)) \Rightarrow (\forall K\ \mathbf{Q2}(K))$

Theorem (G.)

$\mathbf{Q2}(K) \Rightarrow \mathbf{Q1}(K)$

Question (Balka, Elekes, Máthé)

Does **Q2** hold for the Sierpiński carpet?

Tube-null sets

Definition

A set $A \subset \mathbb{R}^2$ is **tube-null** if for every $\varepsilon > 0$, it can be covered by strips of total width at most ε.

Theorem (Pyörälä, Shmerkin, Suomala, Wu, 2020)

The Sierpiński carpet is tube-null.

Theorem (G.)

Let $A \subseteq \mathbb{R}^2$ be a bounded set. If the boundary ∂A is tube-null, then $\mathbf{Q1}(A)$ *and* $\mathbf{Q2}(A)$ *hold.*

The proximal map

Definition

The **proximal map** of a convex function $g : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\operatorname{prox}_g(x) = \operatorname*{arg\,min}_{y \in \mathbb{R}^d} \left(g(y) + \frac{|x-y|^2}{2} \right)$$

($\arg\min$ denotes the unique minimizer).

Proposition

prox_g *is 1-Lipschitz*

Proposition

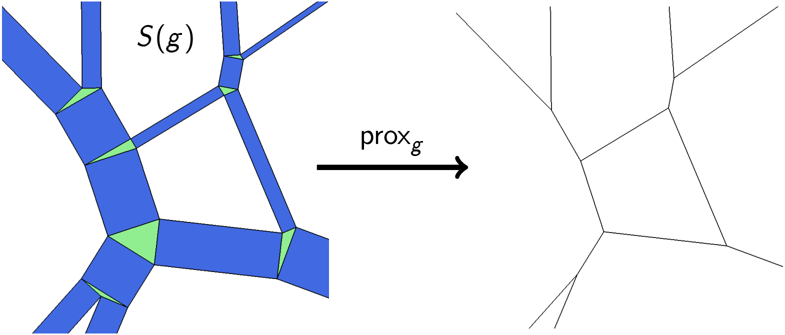
If g is L-Lipschitz, then $|\operatorname{prox}_g(x) - x| \leq L$ *for every* $x \in \mathbb{R}^d$

Generalized strips

Definition

Let $g : \mathbb{R}^d \to \mathbb{R}$ be the maximum of finitely many affine functions. The **generalized strip** associated to g is

$$S(g) = \{x \,:\, g \text{ is not differentiable at } \mathrm{prox}_g(x)\}$$



Generalized strips

Goal: find a generalized strip $S(f)$ covering the tube-null set such that $\operatorname{Lip} f$ is small

Lemma

Let $S(f)$ and $S(g)$ be generalized strips. Then there exists a generalized strip $S(h)$ such that $S(f) \cup S(g) \subseteq S(h)$ and $\operatorname{Lip} h \leq \operatorname{Lip} f + \operatorname{Lip} g$.

Corollary

If K is a compact tube-null set, then K can be covered by some $S(f)$ such that $\operatorname{Lip} f$ is arbitrarily small.

Another application of generalized strips:

Theorem (G.)

Let $A \subseteq \mathbb{R}^d$ be a bounded set. If ∂A can be covered by countably many $\mathcal{C}^2$ hypersurfaces, then $\mathbf{Q1}(A)$ and $\mathbf{Q2}(A)$ hold.

Thank you for your attention!