Equidistribution of cusp points of Hecke triangle groups

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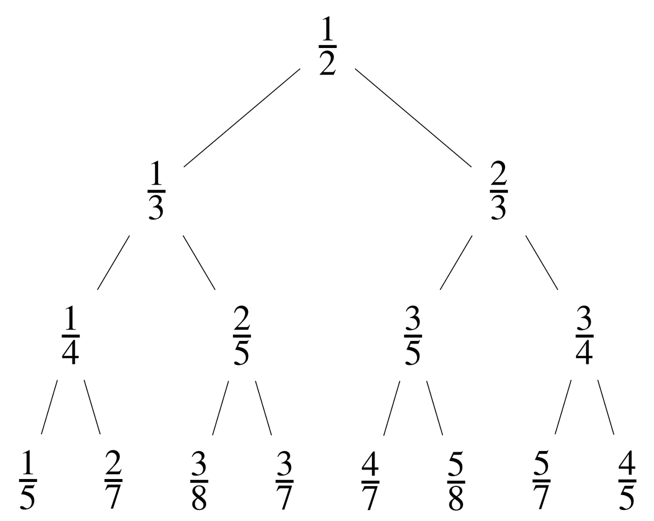
Motivation

The classical Stern–Brocot sequence partitions $\mathbb{Q} \cap [0,1]$ into subsets by constructing iterated mediants starting from $0/1$ and $1/1$.

- Mediant: $\frac{a}{b} \oplus \frac{c}{d} := \frac{a+c}{b+d}$
- Related to $\mathrm{PSL}(2,\mathbb{Z})$ via Farey map



Central question

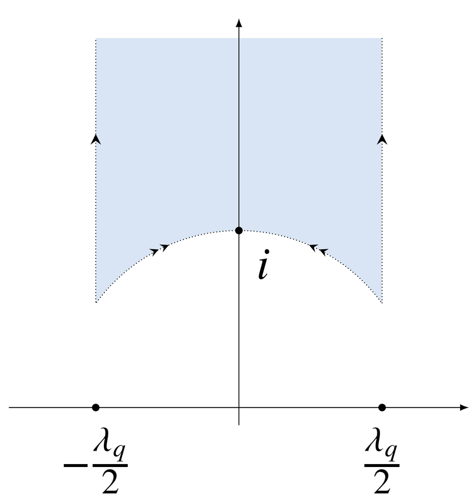
Does arithmeticity of $\mathrm{PSL}(2,\mathbb{Z})$ determine the distribution properties?

Hecke triangle groups

- The Hecke triangle group Γ_q for the parameter $q \in \mathbb{N}_{\geq 3}$ and cusp width $\lambda_q := 2\cos\left(\frac{2\pi}{q}\right)$ is generated by

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T_q := \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}.$$

- $\Gamma_3 = \mathrm{PSL}(2,\mathbb{Z})$ and $\Gamma_3.\infty = \mathbb{Q}$



- $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.x \mapsto \frac{ax+b}{cx+d}.$$

Generalized Farey map

With the geometrically motivated elements

$$g_{q,k} := \left((T_q S)^k S\right)^{-1}$$

in the Hecke triangle group Γ_q, we define the generalized Farey map F_q for odd $q \geq 3$ as the selfmap on $[0,1]$ that is piecewise given by the bijections

$$[g_{q,k}^{-1}.0, g_{q,k}^{-1}.1] \to [0,1], \quad x \mapsto g_{q,k}.x,$$

and

$$[(Qg_{q,k})^{-1}.1, (Qg_{q,k})^{-1}.0] \to [0,1], \quad x \mapsto Qg_{q,k}.x,$$

for $k \in \{(q+1)/2, \ldots, q-1\}$.

- $q = 3$ corresponds to the classical Farey map that generates the Stern–Brocot sequence
- Generalized Stern–Brocot sequence: $S_{-1} := \emptyset$, and for $n \in \mathbb{N}$

$$S_n := F^{-n}(0,1) \setminus S_{n-1}$$

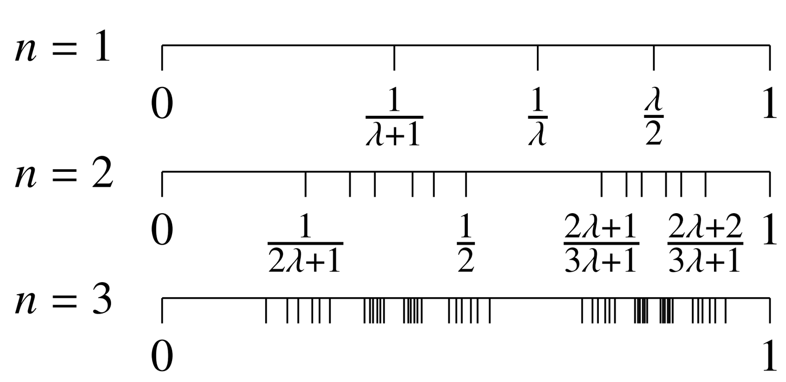


Fig. 1: Stern–Brocot sequence associated to $q = 5$

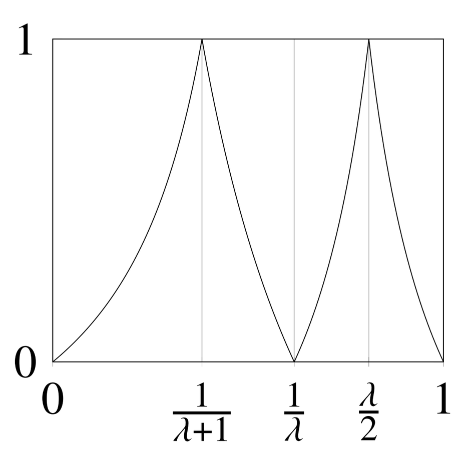


Fig. 2: Graph associated to $q = 5$

- In $\mathbb{Z}[\lambda_q] \times \mathbb{Z}[\lambda_q]$ we identify (a,c) and $(-a,-c)$, and we call the equivalence class (a,c) a reduced fraction if there exists $\begin{bmatrix} a & * \\ c & * \end{bmatrix} \in \Gamma_q$.
- **Lemma ([1]):** The cusp points $\Gamma_q.\infty$ and the reduced fractions in $\mathbb{Z}[\lambda_q] \times \mathbb{Z}[\lambda_q]$ are in bijection.

Distribution results (BKP [1])

We proved that for $q \geq 3$ odd,

(1) For all $0 < \alpha \leq \beta \leq 1$ we have

$$\star\text{-}\lim_{n\to\infty} \left(\log(n) \cdot m|_{F_q^{-n}([\alpha,\beta])}\right) = \log\left(\frac{\beta}{\alpha}\right) m.$$

(2) For each $x \in (0,1]$ we have

$$\star\text{-}\lim_{n\to\infty} x \log(n) \sum_{h \in W_{q,n}} |h'(x)| \delta_{h.x} = m.$$

(3) For each reduced fraction $(v,w) \in \Gamma_q.\infty \cap (0,1]$ we have

$$\star\text{-}\lim_{n\to\infty} c_{v/w}\, vw\, \log(n) \sum_{(r,s)\in \mathrm{RF}_{q,n}(v,w)} \frac{1}{s^2} \delta_{r/s} = m,$$

where $c_1 := 2$ and $c_x := 1$ for $x \neq 1$, and $\mathrm{RF}_{q,n}(v,w)$ denotes the set of reduced fractions in $F_q^{-n}\left(\frac{v}{w}\right)$.

The case $q = 3$ in (1) is established in [2] with a different proof, which however does not apply to $q > 3$.

Takeaway

Not the arithmeticity decides the distribution behavior, but the dynamics and geometry of the underlying group.

Key steps of proof

For all $q \geq 5$:

1. F_q is an AFN-map for the partition ξ of its branches:
   - (A) Adler's condition: The map $(F_q)''/(F_q')^2$ is bounded.
   - (F) Finite image condition: $\{F_q(I) : I \in \xi\}$ is finite.
   - (N) Non-uniformly expanding: $F_q(0) = 0$, $F_q'(0) = 1$, and
   $$|F_q'| \geq \rho(\varepsilon) > 1 \quad \text{on } [\varepsilon, 1].$$
2. F_q is topologically mixing.
3. The transfer operator $(\widehat{F}_q)_m$ of F_q with respect to the Lebesgue measure m,
$$(\widehat{F}_q)_m = \sum_{k=\frac{q+1}{2}}^{q-1} \tau(g_k) + \tau(Qg_k)$$
where
$$\tau(g)f(x) := |(g^{-1})'(x)| f(g^{-1}.x) \quad \text{for } g \in \mathrm{PGL}(2,\mathbb{R}),\ f\colon \mathbb{R}\setminus\{g.\infty\} \to \mathbb{C},$$
has $h(x) = \frac{1}{x}$ as eigenfunction to the eigenvalue 1.

The measure $d\mu = h\, dm$ is σ-finite, infinite, F_q-invariant, ergodic, and conservative.

Define for any $Y \subset [0,1]$ the first return time map

$$\varphi_Y : Y \to \mathbb{N} \cup \{\infty\}, \quad \varphi_Y(x) := \inf\{n \in \mathbb{N} : F^n(x) \in Y\}.$$

For any compact set $C \subset (0,1]$ there exists a measurable set $Y(C) \subset (0,1]$ that contains C and satisfies

$$\mu\left(\{\varphi_{Y(C)} > n\}\right) \sim n^{-1}.$$

With these prerequisites, a result of [3] yields that the iteration of the transfer operators with respect to μ converges uniformly, which we use to finish the proof.

References

[1] L. Breitkopf, M. Kesseböhmer, and A. Pohl. *Equidistribution of cusp points of Hecke triangle groups*. 2024. arXiv: 2402.04784 [math.DS].

[2] M. Kesseböhmer and B. O. Stratmann. A dichotomy between uniform distributions of the Stern-Brocot and the Farey sequence. *Unif. Distrib. Theory* 7.2 (2012), 21–33.

[3] I. Melbourne and D. Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.* 189.1 (2012), 61–110.