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Simultaneous dimension result via transversality

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joint work with Károly Simon and Adam Śpiewak.

Introduction

- Let $\mathcal{S} = \{f_1, \ldots, f_m\}$ be an IFS of contractions on $\mathbb{R}^d$,
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- Let $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$ be the symbolic space and let $\sigma(i_1, i_2, \dots) = (i_2, i_3, \dots)$ the left-shift operator, (denote $\mathbf{i} \wedge \mathbf{j}$ the common part of $\mathbf{i}, \mathbf{j} \in \Sigma$)
- Let $\Pi\colon \Sigma \mapsto \Lambda$ be the natural projection

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$$\Pi(\mathbf{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0),$$

- Let μ be a left-shift invariant, ergodic probability measure on Σ

$$\dim_H \Pi_*\mu = ?, \ \overline{\dim}_P \Pi_*\mu = ?$$

Self-conformal systems

- Suppose that every element $f_i \in \mathcal{S}$ of the IFS is a contracting $C^{1+\alpha}$-conformal mappings (i.e. $D_x f_i \in O(d)$),
- Falconer, Ruelle: For the attractor $\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda)$,

$$\dim_H \Lambda = \overline{\dim}_B \Lambda \leq \min\{d, s_0\},$$

where

$$s_0 \text{ is the unique root of the pressure } P(s) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\mathbf{i}\in\Sigma_n} \|f_{\mathbf{i}}'\|^s\right),$$

where $\Sigma_n = \{1, \dots, m\}^n$ and $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$ for $\mathbf{i} = (i_1, \dots, i_n)$.

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- If Strong Open Set Condition (SOSC) holds

$$\Lambda \cap U \neq \emptyset, \; f_i(U) \cap f_j(U) = \emptyset \text{ and } f_i(U) \subseteq U$$

then $\dim_H \Lambda = s_0$.

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$$\dim_H \Pi_*\mu = \overline{\dim}_P \Pi_*\mu \leq \min\left\{d, \frac{h_\mu}{\chi_\mu}\right\},$$

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$$h_\mu = \lim_{n\to\infty} \frac{-1}{n} \sum_{\mathbf{i}\in\Sigma_n} \mu([\mathbf{i}]) \log \mu([\mathbf{i}]) \text{ and } \chi_\mu = -\int \log \|D_{\Pi(\sigma\mathbf{i})} f_{i_1}\| d\mu(\mathbf{i}).$$

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Question: How can we handle the overlapping cases?

Special case: self-similar systems

- Suppose that $\mathcal{S} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^m$, $|\lambda_i| \in (0,1)$ and $t_i \in \mathbb{R}$.

Special case: self-similar systems

- Suppose that $\mathcal{S} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^m$, $|\lambda_i| \in (0,1)$ and $t_i \in \mathbb{R}$.
- If the Exponential Separation Condition (ESC) holds, i.e.

$$\limsup_{n\to\infty} \frac{1}{n} \log \left(\min_{\mathbf{i}\neq\mathbf{j}\in\Sigma_n} \{ |f_{\mathbf{i}}(0) - f_{\mathbf{j}}(0)| + |\log \lambda_{\mathbf{i}} - \log \lambda_{\mathbf{j}}| \} \right) > -\infty$$

  then

  – Hochman: For every Bernoulli measure $\mu = \{p_1, \ldots, p_m\}^{\mathbb{N}}$,

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  – Shmerkin: $\dim_{L^q} \Pi_*\mu = \min\left\{1, \frac{T(q)}{q-1}\right\}$ for every $1 \neq q > 0$, where

$$\dim_{L^q} \nu = \lim_{r\to\infty} \frac{\log \int \nu(B(x,r))^{q-1} d\nu(x)}{(q-1)\log r} \text{ and } \sum_i p_i^q \lambda_i^{-T(q)} = 1;$$

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  - Jordan and Rapaport: For every ergodic measure $\dim_H \Pi_*\mu = \min\left\{1, \frac{h_\mu}{\chi_\mu}\right\}$.

Special case: rational maps

- Suppose that $\mathcal{S} = \left\{ f_i(x) = \dfrac{a_i x + b_i}{c_i x + d_i} \right\}_{i=1}^{m}$ such that

  $f_i \in C^{1+\alpha}(I),\ f_i(I) \subseteq I$ and $\sup_{x \in I} |f_i'(x)| < 1$ on a compact interval $I \subset \mathbb{R}$.

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- Hochman and Solomyak: If $\mathcal{S}$ satisfies the ESC, i.e.

$$\liminf_{n\to\infty} \frac{1}{n} \log \min_{\substack{i_1 \neq j_1 \\ \mathbf{i},\mathbf{j} \in \Sigma_n}} \sup_{x \in I} \{ |f_{\mathbf{i}}(x) - f_{\mathbf{j}}(x)| \} > -\infty,$$

  then for every Bernoulli measure $\mu = \{p_1, \ldots, p_m\}^{\mathbb{N}}$, $\dim_H \Pi_* \mu = \min \left\{ 1, \dfrac{h_\mu}{\chi_\mu} \right\}$.

Open questions: $\dim_{L^q} \Pi_* \mu =$? & dimension of ergodic measures?

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- Let $I \subset \mathbb{R}^d$ be compact and simply connected. For $\lambda \in U$, let $\mathcal{S}_\lambda = \left\{ f_i^{(\lambda)} \colon I \to I \right\}_{i=1}^{m}$ be a parametrized family of IFS such that $\lambda \mapsto f_i^{(\lambda)}$ is continuous from U to $C^{1+\alpha}(I)$. Let $\Pi_\lambda \colon \Sigma \to I$ be the nat. proj.

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- Suppose that the transversality condition holds, i.e. there exists $C > 0$ such that for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$

$$\eta\left(\{\lambda \in U : \|\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})\| \leq r\}\right) \leq Cr^d.$$

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$$\eta\left(\{\lambda \in U : \|\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})\| \leq r\}\right) \leq C r^d.$$
- Simon, Solomyak and Urbański: For every ergodic measure μ and for η-almost every $\lambda \in U$
$$\dim_H \Lambda_\lambda = \min\{d, s_0(\lambda)\} \text{ and } \dim_H (\Pi_\lambda)_* \mu = \min\left\{1, \frac{h_\mu}{\chi_\mu(\lambda)}\right\}.$$

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- Let $\rho_\lambda\colon \Sigma_* \to \mathbb{R}_+$ be a quasi-multiplicative map depending uniformly continuously on $\lambda \in U$, that is,
  - $\exists C > 0$ and $0 < \alpha < 1$ such that $\rho_\lambda(\mathbf{i}) \le C\alpha^{|\mathbf{i}|}$ for every $\mathbf{i} \in \Sigma_*$ and $\lambda \in U$;
  - $\exists C > 0$ such that $C^{-1} \le \dfrac{\rho_\lambda(\mathbf{ij})}{\rho_\lambda(\mathbf{i})\rho_\lambda(\mathbf{j})} \le C$ for every $\mathbf{i}, \mathbf{j} \in \Sigma_*$ and $\lambda \in U$;
  - $\forall \varepsilon > 0$ and $\lambda_0 \in U$ $\exists \delta > 0$ such that $\rho_\lambda(\mathbf{i})^{1+\varepsilon} \le \rho_{\lambda_0}(\mathbf{i}) \le \rho_\lambda(\mathbf{i})^{1-\varepsilon}$ for every $\mathbf{i} \in \Sigma_*$ and $|\lambda - \lambda_0| < \delta$.

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- Let $\Pi_\lambda\colon \Sigma \mapsto \mathbb{R}^d$ be such that there exists $\log K_n/n \to 0$ for every $\mathbf{i} \ne \mathbf{j} \in \Sigma$
  - $\|\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})\| \le C\rho_\lambda(\mathbf{i} \wedge \mathbf{j})$ for every $\lambda \in U$;
  - $\eta(\{\lambda \in U : \ \|\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})\| \le \rho_\lambda(\mathbf{i} \wedge \mathbf{j})r\}) \le K_{|\mathbf{i}\wedge\mathbf{j}|}r^d$ for every $r > 0$.

Note: This is slightly different than the generalized projection scheme introduced by Solomyak.

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Examples:

- (Self-conformal systems) $\mathcal{S} = \{f_i^{(\mathbf{t})}(x) := f_i(x) + t_i\}_{i=1}^m$ with $f_i \in C^{1+\alpha}([0,1])$ and $(\mathbf{t}) = (t_1, \ldots, t_m) \in U \subset \mathbb{R}^m$ such that $f_i^{(\mathbf{t})}([0,1]) \subset [0,1]$ and $\|f_i'\| < 1/2$, where $\rho_{\mathbf{t}}(\mathbf{i}) = \|(f_{\mathbf{i}}^{(\mathbf{t})})'\|$; (see Simon, Solomyak and Urbański)

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- (Non-autonomous systems) $\Pi_\lambda \colon \{0,1\}^{\mathbb{N}} \mapsto \sum_{n=1}^{\infty} i_n \frac{\lambda^n}{n}$ with $\rho_\lambda(\mathbf{i}) = \lambda^n$; (see Nakajima)

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- (Statistically self-similar systems) For every $\mathbf{i} \in \Sigma_*$, let $X_{\mathbf{i}}$ be i.i.d. compactly supported random variables with abs. cont. density, and let $\lambda_i \in (-1,1) \setminus \{0\}$ and $t_i \in \mathbb{R}$. Then $\Pi_\lambda(\mathbf{i}) = \sum_{n=0}^{\infty}(t_{i_n} + X_{\mathbf{i}|_n})\lambda_{\mathbf{i}|_n}$ with $\rho(\mathbf{i}) = \lambda_{\mathbf{i}}$; (see Jordan, Pollicott, Simon)

A generalized projection scheme

Theorem (B., Simon, Śpiewak). *Under the assumptions above: For η-almost every $\lambda \in U$,*

$$\dim_H (\Pi_\lambda)_* \mu = \min\left\{ d, \frac{h_\mu}{\chi_\mu(\lambda)} \right\} \text{ for every ergodic left-shift invariant measure } \mu.$$

Under the assumptions above: For every ergodic left-shift invariant measure on μ

$$\lim_{n\to\infty} \frac{-1}{n} \log \rho_\lambda(\mathbf{i}|_n) = \lim_{n\to\infty} \frac{-1}{n} \sum_{\mathbf{i}\in\Sigma_n} \mu([\mathbf{i}]) \log \rho_\lambda(\mathbf{i}) =: \chi_\mu(\lambda) \text{ for } \mu\text{-a.e. } \mathbf{i}.$$

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Corollary. *For η-almost every $\lambda \in U$, there exists a unique left-shift invariant ergodic probability measure μ_λ such that*

$$\frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}(\lambda)} = s_0(\lambda), \text{ where } s_0(\lambda) \text{ is the unique root of } P_\lambda(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\mathbf{i}\in\Sigma_n} \rho_\lambda(\mathbf{i})^s \right),$$

and in particular,

$$\dim_H \Pi_\lambda(\Sigma) = \max\{\dim_H (\Pi_\lambda)_* \nu : \nu \text{ is ergodic}\} = \dim_H (\Pi_\lambda)_* \mu_\lambda = \min\{d, s_0(\lambda)\}.$$

A few words on the proof

- An ergodic left-shift invariant measure μ is called *k-step Markov* if for every $n \geq k$ and $\mathbf{i} = (i_1, \ldots, i_n) \in \Sigma_*$
$$\nu([i_1, \ldots, i_n]) = \nu([i_1, \ldots, i_k]) \prod_{j=1}^{n-k} P_\nu(i_{j+k} | i_j, \ldots, i_{j+k-1}),$$
where
$$P_\nu(i | j_1, \ldots, j_k) = \begin{cases} \dfrac{\nu([j_1, \ldots, j_k, i])}{\nu([j_1, \ldots, j_k])} & \text{if } \nu([j_1, \ldots, j_k]) > 0, \\ 0 & \text{if } \nu([j_1, \ldots, j_k]) = 0. \end{cases}$$

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- For two ergodic shift-invariant measures μ and ν such that $\mu([\mathbf{i}]) > 0 \Rightarrow \nu([\mathbf{i}]) > 0$ for every $\mathbf{i} \in \Sigma_*$, let
$$h(\mu \| \nu) = \lim_{n\to\infty} \frac{1}{n} \sum_{\mathbf{i}\in\Sigma_n} \mu([\mathbf{i}]) \log \frac{\mu([\mathbf{i}])}{\nu([\mathbf{i}])} \text{ be the Kullback-Liebler divergence,}$$
if the limit exists.
- Note: $h(\mu\|\nu) \geq 0$ and $h(\mu\|\nu) = 0$ if and only if $\mu = \nu$; and it exists if ν is a k-step Markov.

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Lemma. *There exists a countable subset* $\mathcal{D}$ *of ergodic left-shift invariant measures such that every* $\nu \in \mathcal{D}$ *is a* k*-step Markov for some* $k \in \mathbb{N}$*, moreover, for every* $\varepsilon > 0$ *and every* μ *ergodic there exists a* $\nu \in \mathcal{D}$ *such that*

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- For $\nu \in \mathcal{D}$, let $V(\nu, \varepsilon) = \{\mu \text{ ergodic}: \text{ the four property above holds}\}$,
- For $\lambda_0 \in U$, let $U' \subset U$ such that $\rho_{\lambda_0}(\mathbf{i})^{1+\varepsilon} \leq \rho_\lambda(\mathbf{i}) \leq \rho_{\lambda_0}(\mathbf{i})^{1-\varepsilon}$ for $\mathbf{i} \in \Sigma_*$

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- For $\nu \in \mathcal{D}$, let $V(\nu, \varepsilon) = \{\mu \text{ ergodic} : \text{ the four property above holds}\}$,
- For $\lambda_0 \in U$, let $U' \subset U$ such that $\rho_{\lambda_0}(\mathbf{i})^{1+\varepsilon} \leq \rho_\lambda(\mathbf{i}) \leq \rho_{\lambda_0}(\mathbf{i})^{1-\varepsilon}$ for $\mathbf{i} \in \Sigma_*$

Proposition. *For* η*-almost every* $\lambda \in U'$*, for every* $\mu \in V(\nu, \varepsilon)$

$$\dim_H (\Pi_\lambda)_* \mu \geq \min\left\{d, \frac{h_\mu}{\chi_\mu(\lambda)}\right\} - O(\varepsilon).$$

The theorem follows by a standard covering and density argument.

A few words on the proof

For every $\mu \in V(\nu, \varepsilon)$, let A_μ be such that $\mu(A_\mu) > 1 - \delta$, where

$$A_\mu = \{\mathbf{i} \in \Sigma : \mu([\mathbf{i}|_n]) \leq C_\mu e^{-n(h_\nu - \varepsilon)},\ C_\mu^{-1} e^{-n(\chi_\mu(\lambda_0)+\varepsilon)} \leq \rho_{\lambda_0}(\mathbf{i}|_n) \\ \text{and } \mu([\mathbf{i}|_n]) \leq C_\mu e^{\varepsilon n} \nu([\mathbf{i}|_n])\}.$$

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$$\int_{U'} \sup_{\mu \in V(\nu,\varepsilon) \cap \{C_\mu < C\}} \iint \frac{d\mu|_{A_\mu}(\mathbf{i}) d\mu|_{A_\mu}(\mathbf{j})}{\|\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})\|^s} d\eta(\lambda)$$

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Exponential Distance from the Enemy (EDE)

- For a fixed $\lambda_0 \in U$, an infinite sequence $\mathbf{i} \in \Sigma$ has exponential distance from the enemy if for every $\varepsilon > 0$ there exists $C > 0$ such that

$$\operatorname{dist}\left(\Pi_{\lambda_0}(\mathbf{i}), \bigcup_{\mathbf{j}\in\Sigma_n : \mathbf{j}\neq\mathbf{i}|_n} \Pi_{\lambda_0}([\mathbf{j}])\right) > C\rho_\lambda(\mathbf{i}|_n)e^{-\varepsilon n} \text{ for every } n \in \mathbb{N}.$$

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Theorem (B., Simon, Śpiewak). *Suppose that $s_0(\lambda) < d$ for every $\lambda \in U$. Then for η-almost every $\lambda \in U$ the following holds: for every ergodic measure μ, μ-almost every $\mathbf{i}$ has exponential distance from the enemy.*

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Corollary. *Under the conditions above, for η-almost every $\lambda \in U$ the following holds: For every ergodic measure μ, there exists a set $A \subset \Sigma$ such that $\mu(A) = 1$ and $\Pi_\lambda|_A$ is invertible and its inverse is Hölder continuous for every Hölder exponent $\alpha \in (0, 1)$.*

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$$d_{(\Pi_\lambda)_*\nu}(\Pi_\lambda(\mathbf{i})) = \frac{h(\nu\|\mu) - h_\nu}{\chi_\mu(\lambda)} \text{ for } \mu\text{-almost every } \mathbf{i},$$

where $h(\nu|\mu) = \lim_{n\to\infty} \frac{-1}{n} \sum_{\mathbf{i}\in\Sigma_n} \mu([\mathbf{i}]) \log \nu([\mathbf{i}])$.

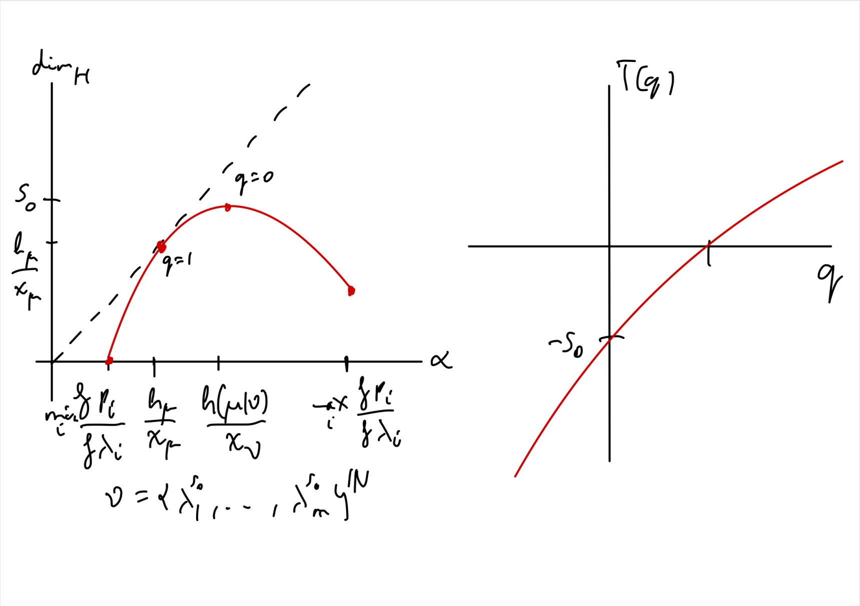
An application: Multifractal analysis

Question Given a measure ν, describe the map $\alpha \mapsto \dim_H \{x : d_\nu(x) = \alpha\}$, where

$$d_\nu(x) = \lim_{r\to\infty} \frac{\log \nu(B(x,r))}{\log r}.$$

An application: Multifractal analysis

- Arbeiter and Patschke: For the self-similar IFS $\mathcal{S} = \{f_i(x) = \lambda_i O_i x + t_i\}_{i=1}^m$ with SOSC, if $\mu = \{p_1, \ldots, p_m\}^{\mathbb{N}} \neq \{\lambda_1^{s_0}, \ldots, \lambda_m^{s_0}\}^{\mathbb{N}}$ is a Bernoulli measure then
$$\dim_H \{x : d_{\Pi_* \mu}(x) = \alpha\} = \inf_{q \in \mathbb{R}} (q\alpha - T(q)) \text{ for } \alpha \in I = \left[\min_i \frac{\log p_i}{\log \lambda_i}, \max_i \frac{\log p_i}{\log \lambda_i}\right],$$
where $\sum_i p_i^q \lambda_i^{-T(q)} = 1$, and $\{x : d_{\Pi_* \mu}(x) = \alpha\} = \emptyset$ for $\alpha \notin I$.



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- Barral and Feng: For the self-similar IFS $\mathcal{S} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^m$ with ESC on $\mathbb{R}$ and $\sum_i |\lambda_i| < 1$, if $\mu = \{p_1, \ldots, p_m\}^{\mathbb{N}} \neq \{\lambda_1^{s_0}, \ldots, \lambda_m^{s_0}\}^{\mathbb{N}}$ then
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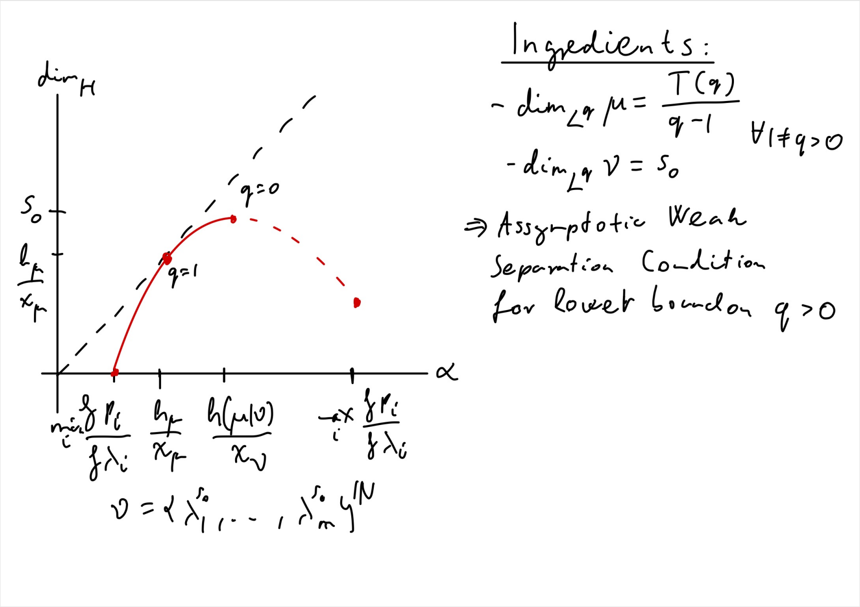

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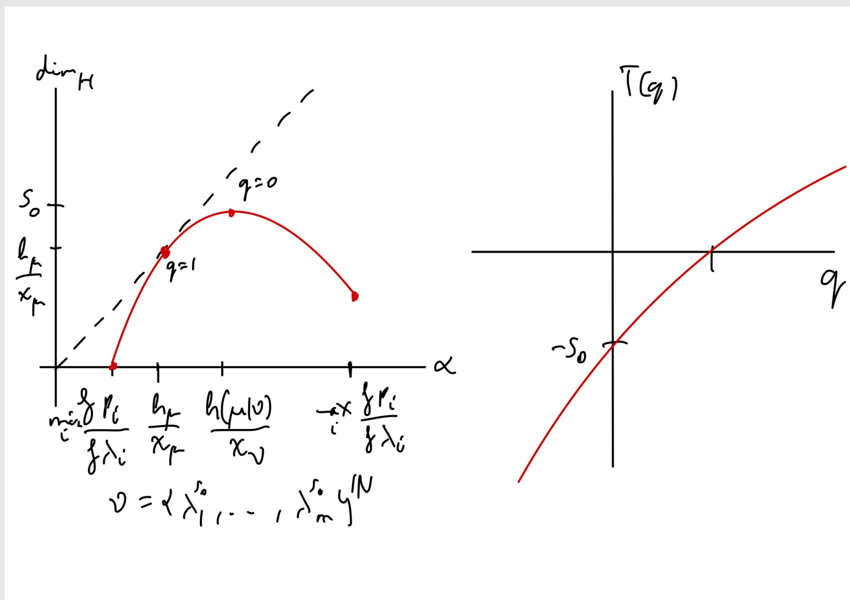

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- Enough for $q < 0$.
- $\dim_H \{x : d_{\Pi_* \mu}(x) = \alpha\} \leq \inf_{q<0} (q\alpha - T(q))$ by covering for $\sum_{i=1}^m p_i^q \lambda_i^{-T(q)} = 1$.

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- $\dim_H \{x : d_{\Pi_*\mu}(x) = \alpha\} \geq \sup \left\{ \frac{h_\nu}{\chi_\nu} : \frac{h(\nu \| \mu) - h_\nu}{\chi_\nu} = \alpha \right\} = \inf_{q<0} (q\alpha - T(q))$.

Thank you for your attention!