Homogeneous non-Rajchman self-similar measures on the line

Tianhan Yi

The Chinese University of Hong Kong

Geometry and Fractals under the Midnight Sun

University of Oulu

(Based on joint work with De-Jun Feng)

Fourier decay for measures

- μ is a finite Borel measure on $\mathbb{R}$.
- Fourier transform

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi x} d\mu(x).$$

- μ is **Rajchman** if $\lim_{\xi \to \infty} \hat{\mu}(\xi) = 0$.

Self-similar measures

- Let $m \geq 2$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (0,1)^m$, $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{p} = (p_1, \ldots, p_m)$ a probability measure.
- $\mu = \mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is **self-similar** associated with the IFS $\{\varphi_j\}$ and the probability $\mathbf{p}$ if

$$\mu = \sum_{j=1}^{m} p_j \mu \circ (\varphi_j^{-1}), \qquad \text{where } \varphi_j(x) = \lambda_j x + a_j.$$

- $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is called to be **homogeneous** if $\lambda_j = \lambda$ for all $j = 1, \ldots, m$.

Problem: Can we characterize Rajchman self-similar measures on the line?

Homogeneous case

If $m = 2$, $\lambda_1 = \lambda_2 = \lambda$ and $p_1 = p_2 = \frac{1}{2}$, then $\mu_\lambda = \mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is the Bernoulli convolution associated with λ.

Theorem (Erdös 1939)

If λ^{-1} is a Pisot number not equal to 2, then μ_λ is not Rajchman.

Theorem (Salem 1943)

If μ_λ is not Rajchman, then λ^{-1} is Pisot.

Remark: When $m \geq 3$ and $\mathbf{a} \in \mathbb{Z}^m$, their methods still work after a slight modification (see e.g. Lau-Ngai-Rao 2001).

Non-homogeneous case

Assume $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is a non-trivial self-similar measure with $\mathbf{p} > 0$.

Theorem (Li-Sahlsten 2022)

If there exist $i \neq j$ such that $\log \lambda_i / \log \lambda_j \notin \mathbb{Q}$, then $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is Rajchman for all $\mathbf{a}$ and $\mathbf{p}$.

Theorem (Brémont 2021)

Let $\lambda_j = \lambda^{n_j}$ for $n_j \in \mathbb{N}$ with $\gcd(n_1, \ldots, n_m) = 1$. There exists some $\mathbf{p}$ such that $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is not Rajchman if and only if λ^{-1} is Pisot and the IFS is linearly conjugate to one with $a_j \in \mathbb{Q}(\lambda)$.

Remark: Rapaport extends Brémont's result to high dimension.

Main result

Question: In homogeneous case, if the IFS is of Pisot type as in Brémont's setting, is $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ non-Rajchman for all $\mathbf{p}$?

Theorem (Feng-Y, 2024)

Let $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ be a homogeneous self-similar measure. If λ^{-1} is a non-integer Pisot number and $a_j \in \mathbb{Q}(\lambda)$, then $\mu^{\mathbf{p}}_{\boldsymbol{\lambda},\mathbf{a}}$ is not Rajchman for all $\mathbf{p}$.

Remark

- Our proof is inspired by Erdös' argument. The difficulty is when $a_j \in \mathbb{Q}(\lambda)$, the distribution of the roots of the trigonometric polynomial associated with the self-similar measure is much more complicated.
- When the IFS is non-homogeneous or it is homogeneous but $\lambda^{-1} \in \mathbb{Z}$, it is possible that the measure is Lebesgue.

THANK YOU