Topological expansive Lorenz maps with a hole at the critical point

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Geometry and Fractals under the Midnight Sun
University of Oulu
27 June 2024

Outline

- Background

- Main results

1. Background

Open dynamical systems

- Discrete dynamical system (X, f), where $X = [0, 1]$ and $f : X \to X$ is continuous with positive topological entropy. Let $H \subseteq X$ be a connected subinterval, called the hole. Survivor set:

$$S_f(H) = \{x \in X : f^n(x) \notin H \ \forall n \geq 0\} = X \setminus \bigcup_{n=0}^{\infty} f^{-n}(H).$$

- The survivor set $S_f(H)$ depends on (1) the map f, (2) the size of H and (3) the position of H.

Devil's staircase

- Urbański (1986): Let $f = T_2 = 2x \pmod 1$ with a hole $(0, t)$. Then

$$\eta_2 : t \mapsto \dim_{\mathcal{H}} S_2(t)$$

is a devil's staircase, that is, η_2 is decreasing, and η_2 is constant Lebesgue almost everywhere.

- Kalle, Kong, Langeveld & Li (2020): Let $f = T_\beta(x) = \beta x \pmod 1$ with a hole $(0, t)$, where $\beta \in (1, 2]$ be fixed. The dimension function

$$\eta_\beta : t \mapsto \dim_{\mathcal{H}} S_\beta(t)$$

is a devil's staircase.

- Langeveld & Samuel (2023): Let $f = T_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$ with a hole $(0, t)$, where $\beta \in (1, 2]$ and $\alpha \in (0, 2 - \beta)$ be fixed. Then

$$\eta_{\beta,\alpha} : t \mapsto \dim_{\mathcal{H}} S_{\beta,\alpha}(t)$$

is a devil's staircase.

Devil's staircase

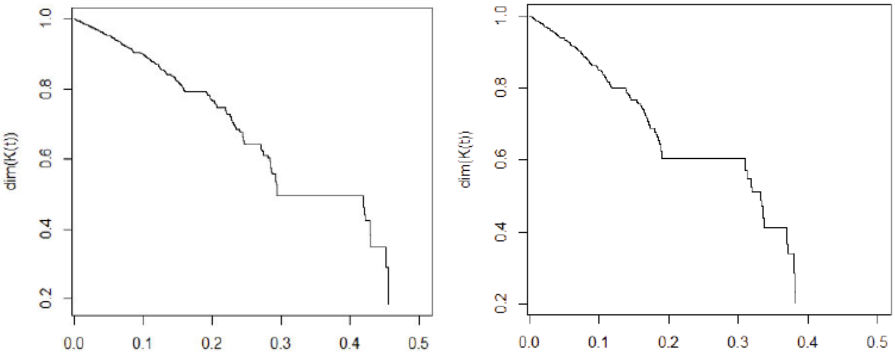

Figure: Left: $\eta_\beta : t \mapsto \dim_{\mathcal{H}} S_\beta(t)$ where β is tribonacci number.
Right: $\eta_{\beta,\alpha} : t \mapsto \dim_{\mathcal{H}} S_{\beta,\alpha}(t)$ where β is golden mean and $\alpha = 1 - \beta/2$.

Questions

A Lorenz map on $X = [0,1]$ is an interval map $f : X \to X$ with a critical point $c \in (0,1)$ such that
(i) f is strictly increasing on $[0,c)$ and on $(c,1]$;
(ii) $\lim_{x\uparrow c} f(x) = 1$, $\lim_{x\downarrow c} f(x) = 0$.
f is called expansive Lorenz map if additionally the preimage set $C(f) = \cup_{n\geq 0} f^{-n}(c)$ of c is dense in X.

Questions:

1. Can we extend the results from $T_{\beta,\alpha}$ to expansive Lorenz maps?
2. If we consider the hole near c, is there any relationship between the holes $(0,t)$ and $(a,b) \ni c$? Can we still have the devil's staircase?

2. Main results

Main results

Denote ELM as the set of expansive Lorenz maps. Let
$S_f(a,b) := \{x \in [0,1] : f^n(x) \notin (a,b), \forall n \geq 0\}$, and
$S_f^+(a,b) := \{x \in [0,1] : f(b) \leq f^n(x) \leq f(a)\ \forall n \geq 0\}$.

Proposition (S.-Li-Ding, 2024)

1. $S_f(a,b) \setminus S_f^+(a,b)$ is countable and
$$h_{top}(f|S_f(a,b)) = h_{top}(f|S_f^+(a,b)).$$
2. Let $f \in ELM$ with a hole (a,b). Then the case $a = c$ is equivalent to the hole $(0, f(b))$.

Main results

Theorem 1 (S.-Li-Ding, 2024)

Let $f \in ELM$ with a hole (a, b). If $\tau_f(b+)$ is periodic, then there exists a maximal plateau $I(b)$ such that for all $b' \in I(b)$, $S_f^+(a, b') = S_f^+(a, b)$. The endpoints of $I(b)$ are also characterized.

Theorem 2 (S.-Li-Ding, 2024)

Let $f \in ELM$ with ergodic a.c.i.m.. Then the entropy function

$$\lambda_f(a) : b \mapsto h_{top}(f | S_f(a, b))$$

is a devil's staircase, where a is fixed.

Corollary (S.-Li-Ding, 2024)

Let $f = T_{\beta,\alpha}$ with a hole (a, b) where $a \in [0, \frac{1-\alpha}{\beta}]$ is fixed. Then the dimension function $\eta_f(a) : b \mapsto \dim_{\mathcal{H}}(S_f(a, b))$ is a devil's staircase.

Future work

Question: If we consider the case $(a, b) \subsetneq (0, c)$, can we still obtain the devil's staircase with a being fixed? The problem is that the tools from Lorenz shifts cannot be used directly.

References:

1. Urbánski M. On Hausdorff dimension of invariant sets for expanding maps of a circle *Ergod. Th. & Dynam. Sys.* 6 295-309 (1986)
2. Kalle C, Kong D, Langeveld N and Li W. The β-transformation with a hole at 0 *Ergod. Th. & Dynam. Sys.* 40 2482-2514 (2020)
3. Langeveld N and Samuel T, Intermediate β-shifts as greedy β-shifts with a hole *Acta Math. Hungar.* 170 269-301 (2023)
4. Sun Y, Li B and Ding Y. Topological Expansive Lorenz Maps with a Hole at Critical Point *J. Stat. Phys.* 191 50 (2024)

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Thanks for your attention!